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ABSTRACT

The purpose of this study was to determine the effect of the TI-81 Graphics Calculator as an instructional tool on college algebra students' conceptual and procedural achievements involving functions and graphs and on the students' attitudes toward mathematics. The study was a posttest-only, equivalent control group design conducted with students enrolled in four sections of college algebra in a two-year technical college. Each student in the treatment classes was issued a TI-81 Graphics Calculator for the duration of the semester. Results provided evidence that the use of a graphics calculator as a learning tool had a significant effect on college algebra students' performance on procedures involving functions and graphs. Statistically, the mean attitude scores were the same for the treatment and control groups. Appendix includes classroom handouts. (MKR)

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Effect of Graphics Calculators On College Students' Learning of Mathematical Functions and Graphs

by
Frank W. Caldwell, Jr.

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EFFECT OF A GRAPHICS CALCULATOR ON COLLEGE STUDENTS' LEARNING OF MATHEMATICAL FUNCTIONS AND GRAPHS

The introduction of functions and graphs is a critical moment in mathematics education since it presents a setting with the opportunity for powerful learning to take place and since the concept of functions and graphs is fundamental to more sophisticated parts of mathematics (Leinhardt, Zaslavsky, and Stein, 1990). High school students experience a major leap in their mathematical development when they are introduced to the concept of the graph of a function in two variables (Herscovics, 1989). Results from the Second Mathematics Assessment of the National Assessment of Educational Progress (Carpenter, Corbitt, Kepner, Linquist, and Reys, 1981) indicate that the majority of students do not manage this leap. Carpenter et al. found that students could graph ordered pairs of numbers in the Cartesian plane but that most did not understand the relationship between equations and their graphs. This same assessment indicated that students had difficulties with function notation and the formalization of a known functional relationship between two variables into an algebraic expression.

A report from the Second International Mathematics Study (McKnight, Travers, and Dossey, 1985) indicated that precalculus and calculus students performed poorly in the content area of elementary functions. Results from the Fourth National Assessment of Educational Progress (Brown et al., 1988; Silver et al., 1988) indicated that U.S. students had a limited understanding of function concepts and of graphing. Brown et al. (1988) reported that secondary students demonstrated some intuitive knowledge of functions and also demonstrated some degree of facility in computing specific values of functions but experienced difficulty with items covering a broad range of functional concepts.

Kenelly (1986) contends that calculus students experience difficulty with function concepts. He surmises that beginning algebra students fail to form a

conceptual understanding of variables; as a result, for many students variables are simply symbols used in manipulative practice exercises, and functions are "ordered pairs of these things". Students miss the idea that functions capture the spirit and essence of connections and interdependencies, and they fail to see that functions embrace the elements of input and output, control and observation, and cause and effect. Epps (1986) says that beginning calculus students do not know the abstract definition of graph of a function. The can plot and connect points but they do not know that for a function f, f(x) is the height to the graph of f at x.

Technology has the potential to enhance the understanding of functions and graphs, but this potential has not been fully realized. The hand-held scientific calculator was introduced in the 1970s, but it has not had a significant impact on the teaching and learning of mathematics in the United States. The microcomputer has not had the impact on mathematics education that had been predicted (Barrett and Goebel, 1990) since many schools do not have a computer in each mathematics classroom and since many educators have trouble defining the role of the computer in the classroom. Most algebra teachers indicate that they use computers for demonstration purposes only (Demana and Waits, 1992). The computer has had a great impact on mathematics, but mathematics is still being taught in most college courses just as it was 30 years ago as a paper-and-pencil discipline (National Research Council, 1991).

In recent years much attention has been focused on the reform of mathematics education with the aid of technology. Mokros and Tinker (1987) conducted studies to determine how middle school students learn graphing skills through microcomputer-based laboratories. Scores on graphing items indicated a significant improvement in students' ability to interpret and use graphs between pretests and posttests when using microcomputer-based laboratory units.

Ganguli (1990) conducted a study to investigate the effect of the microcomputer as a demonstration tool on the achievements and attitudes of college students enrolled in an intermediate algebra class in which two classes were taught selected topics with teacher-demonstrated microcomputer graphs and two classes were taught the same selected topics with graphs drawn by the teacher on the chalkboard. After completion of five weeks of instruction, a 16-item multiple choice posttest was administered; at the end of the quarter, a two-hour comprehensive examination was administered. The treatment effect was significant for the comprehensive examination but not for the posttest. Ganguli concluded that the significant difference in the final examination indicated that students had acquired and retained conceptualizations of algebra better in the treatment group than in the control group.

Rich (1990/1991) conducted a study to examine the effects of graphing calculators on precalculus students' achievements, attitudes, and problem-solving approaches in which two classes were taught precalculus using Casio fx-7000G graphing calculators and six classes were taught precalculus without graphing calculators. The study did not provide evidence of an overall achievement effect of graphing calculator instruction. Rich concluded that students taught precalculus with the use of graphing calculators acquire a better understanding of the relationship between an algebraic equation and its graph, and the use of graphing calculators encourages student exploration and conjecturing.

Giamati (1990/91) conducted a study to investigate how graphing calculators would improve students' understanding of how variations in an equation affect the transformation of a graph. In two experimental precalculus classes students used graphing calculators in pairs with worksheets designed by the teacher. Two control precalculus classes did not use graphing calculators. Two posttests were administered at the end of the study. The means of the first posttest did not differ

significantly when analyzed with a one factor ANOVA, but the mean of the calculator group was lower than the mean of the control group. There was a significant difference between the means of the experimental and control groups on the second posttest, and the mean of the experimental group was lower than the mean of the control group. Giamati found that students with poor conceptual links between equations and graphs were impeded by the use of a graphing calculator. It appeared that students needed a solid understanding of the concepts of graphing before graphing tools could be beneficially incorporated into instruction.

Estes (1990) conducted a study to investigate the effects of implementing graphing calculators and computer technologies as teaching tools in Applied Calculus. The researcher designed and taught two experimental classes in Applied Calculus with the use of Casio fx-7000G graphing calculators and a Macintosh SE computer with overhead projector capabilities. Special assignments were developed and used to facilitate learning the concepts of Applied Calculus with the graphing calculator. Students in a control group were allowed to use calculators, but did not receive any special attention to effective uses of the graphing calculator to facilitate conceptual learning. At the end of the semester, a 25-question multiple choice test compiled from available AB Calculus tests was administered to both the experimental and control groups. Of these questions, 18 were procedural and 7 were conceptual. The experimental group scored significantly higher than the control group on conceptual measures. No significant difference was found between the groups on the procedural measures.

These studies yielded inconclusive evidence of the value of technology in mathematics education. The current study investigated the effect of integrating the TI-81 Graphics Calculator into a college algebra class on students'

understanding of concepts, performance of procedures, and attitudes toward mathematics.

METHOD

This study was a posttest-only, equivalent, control group design conducted with students enrolled in four sections of college algebra in a two-year technical college. Students registered for two of the sections to be taught simultaneously were randomly assigned to the two sections. It was determined by random assignment which section would be the treatment class and which section would be the control class. The researcher and a second instructor were randomly assigned to teach the two sections. The same assignment procedure was used for the other two college algebra sections which were taught simultaneously but at a different hour than the first two sections. The researcher and the second instructor were assigned classes for the second two sections so that both the researcher and the second instructor each taught one treatment class and one control class.

Each student in the treatment classes was issued a TI-81 Graphics Calculator for the duration of the semester. Students in the treatment classes used the TI-81 Graphics Calculators for classroom learning activities, for assignments outside of class, and for tests and examinations. The TI-81 Graphics Calculators were used for performing calculations; for graphing functions and relations; and for solving equations, inequalities, systems of equations, and systems of inequalities. The instructors in the treatment classes used overhead projector models of the TI-81 Graphics Calculator for demonstration purposes. Instruction in the use of the TI-81 Graphics Calculator was integrated into the college algebra course by means of handouts and instructor-conducted demonstrations and explanations.

Students in the control classes used ordinary scientific calculators without graphing capabilities for performing calculations and coordinate graph paper for drawing graphs. The instructors in the control classes used ordinary scientific calculators for performing calculations and coordinate chalkboards for drawing graphs. The students and instructors in the control classes used traditional paper-and-pencil techniques with the aid of ordinary scientific calculators for solving equations, inequalities, systems of equations, and systems of inequalities.

The researcher constructed 13 classroom-learning activities employing the TI-81 Graphics Calculator and covering various topics on functions and graphs. The 13 classroom-learning activities were designed to guide students in the treatment classes by leading questions to discover patterns and make conjectures about functions and graphs. The learning activities featured procedural guidelines for using graphics calculators, drill and practice, development of vocabulary, leading questions, formation of conjectures, and verification of conjectures. Some of the classroom-learning activities are appended to this paper.

Researcher-constructed posttests were administered to both treatment and control classes at the end of the study. One 22-item instrument measured students' conceptual achievements involving functions and graphs. The use of calculators was not permitted on this test. A second 22-item instrument measured students' procedural achievements involving functions and graphs. The treatment classes used graphing calculators on this test, and the control classes used ordinary scientific calculators. A third 25-item instrument measured students' attitudes toward mathematics. Test scores for treatment and control classes were compared using analysis of variance with a level of significance of 0.05.

RESULTS

The concepts test included the following concepts: function, domain, range, symmetry, increasing function, decreasing function, inverse function, translation, and intercept. The mean, standard deviation, minimum, and maximum of the concepts-test scores are illustrated in Table 1. A one-way analysis of variance on the data is illustrated in Table 2. The mean for the treatment group is lower than the mean for the control group. The analysis of variance does not indicate a significant difference between the two groups.

Table 1. Mean, standard deviation, minimum, and maximum of concepts-test scores.

Group	N	Mean	Standard Deviation	Minimum	Maximum
Treatment	41	12.927	4.655	5.000	22.000
Control	39	13.385	3.704	6.000	21.000

Table 2. Analysis of variance of concepts-score data.

Source of Variance	DF	Sum of Squares	Mean Square	F Value	Pr > F
Between groups	1	4.189	4.189	0.24	0.6289
Within groups	78	1388.011	17.795		
Total	79	1392.200			

The procedures test included the following procedures: determining slope of a line, domain and range of a function, type(s) of symmetry, zero(s) of a function, intercept(s) of a graph, interval(s) of increase and decrease of a function, composition of functions, and inverse of a function; matching of function and graph; and solving equations. The mean, standard deviation, minimum, and maximum of the procedures-test scores are illustrated in Table 3. A one-way analysis of variance on the data is illustrated in Table 4. The mean for the treatment group is higher than the mean for the control group. The analysis of variance indicates a significant difference between the two groups at a level of significance of 0.05.

Table 3. Mean, standard deviation, minimum, and maximum of concepts-test scores.

Group	N	Mean	Standard Deviation	Minimum	Maximum
Treatment	41	12.073	4.834	4.000	22.000
Control	39	9.513	4.058	3.000	19.000

The attitude survey was developed to measure a student's general attitude toward mathematics. The survey was designed to collectively look at the following aspects of a student's attitude toward mathematics: the student's position relative to the importance of mathematics, the usefulness of mathematics, and the need to study mathematics; the student's feelings about the use of calculators in mathematics; the student's concept of what mathematics really involves; and the

student's feelings about various mathematical activities. This was a Likert-type survey. For the positively-stated items, a value of five was assigned to each "strongly agree" response, a value of four to each "agree" response, a value of three to each "undecided" response, a value of two to each "disagree" response, and a value of one to each "strongly disagree" response. The assignment of values was reversed for each negatively-stated item.

Table 4. Analysis of variance of procedures-test data.

Source of Variance	DF	Sum of Squares	Mean Square	F Value	Pr > F
Between groups	1	131.026	131.026	6.55	0.0124
Within groups	78	1560.524	20.007		
Total	79	1691.550			

The mean, standard deviation, minimum, and maximum of the attitude-survey scores are illustrated in Table 5. A one-way analysis of variance on the data is illustrated in Table 6. The mean for the treatment group is lower than the mean for the control group. The analysis of variance does not indicate a significant difference between the two groups.

The analysis of attitude-survey scores for the treatment group identified the two lowest scores as outliers. A second analysis of the attitude-survey scores was performed with these two outliers omitted. The mean, standard deviation, minimum, and maximum of the attitude-survey scores with the omission of the

outliers is illustrated in Table 7. A one-way analysis of variance on the data with the omission of the outliers is illustrated in Table 8. The mean for the treatment group is higher than the mean for the control group. The analysis of variance does not indicate a significant difference between the two groups.

Table 5. Mean, standard deviation, minimum, and maximum of attitude-survey scores.

Group	N	Mean	Standard Deviation	Minimum	Maximum
Treatment	41	88.805	10.472	59.000	102.000
Control	39	89.974	8.746	74.000	106.000

Table 6. Analysis of variance of attitude-survey data.

Source of Variance	DF	Sum of Squares	Mean Square	F Value	Pr > F
Between groups	1	27.337	27.337	0.29	0.5903
Within groups	78	7293.413	95.505		
Total	79	7320.750			

Table 7. Mean, standard deviation, minimum, and maximum of attitude-survey scores with the omission of outliers.

Group	N	Mean	Standard Deviation	Minimum	Maximum
Treatment	39	90.333	8.141	72.000	102.000
Control	39	89.974	8.746	74.000	106.000

Table 8. Analysis of variance of attitude-survey data with the omission of outliers.

Source of Variance	DF	Sum of Squares	Mean Square	F Value	Pr > F
Between groups	1	2.513	2.513	0.04	0.8517
Within groups	76	5425.641	71.390		
Total	77	5428.154			

DISCUSSION

The purpose of this study was to determine the effect of the TI-81 Graphics Calculator as an instructional tool on college algebra students' conceptual and procedural achievements involving functions and graphs. This study yielded inconclusive evidence of the value of the graphics calculator. The study provided evidence that the use of a graphics calculator as a learning tool had a significant effect on college algebra students' performance of procedures involving functions and graphs. The treatment group, which used graphics calculators, exhibited a

higher level of achievement than the control group. The study did not provide evidence that the use of a graphics calculator as a learning tool had a significant effect on college students' understanding of concepts involving functions and graphs. The control group exhibited a higher level of achievement than the treatment group. The study did not provide evidence that the use of a graphics calculator had a significant effect on college algebra students' attitudes toward mathematics. Statistically, the mean attitude scores were the same for the treatment group and the control group. However, the control group had a higher mean attitude score in the first analysis of attitude-survey scores, and the treatment group had a higher mean attitude score in the second analysis of attitude-survey scores with the omission of two outliers from the treatment group.

Several revelant observations were made during the study. Use of graphics calculators requires student understanding and skill in the use of algebraic logic. Since calculators must be given precise instructions, students must understand and become proficient in the use of algebraic symbols. To evaluate an expression with the TI-81 Graphics Calculator, the student must enter the expression correctly. This often involves the use of nested parentheses, exponents, roots, and reciprocals. The use of the graphics calculator requires algebraic manipulative skills. In order to graph functions and relations with the TI-81 Graphics Calculator, the student must first solve the equation for y and then enter the resulting expression in the Y= list. To produce useful graphs of functions and relations, students need algebraic estimation skills. Students must be able to determine reasonable domains and ranges for functions and relations and appropriate scales for the axes.

The researcher discovered that it takes classroom time to teach students to use graphics calculators. Some of the time can be reclaimed by using graphics calculators to aid with such tedious and time-consuming procedures as evaluating

functions, solving excessively-complicated equations and inequalities, and plotting graphs.

Students in the treatment group used hands-on classroom-learning activities constructed by the researcher which employed the TI-81 Graphics Calculator. Learning activities were designed to guide students by leading questions to discover and investigate patterns and to form and verify conjectures. This learning method was new for nearly all of the students in the treatment group who were comfortable with the traditional lecture mode. Many students were frustrated by the learning activities because they had difficulty forming and verifying conjectures.

The researcher discovered that it takes less time and effort to teach concepts and procedures with the traditional lecture mode than to have students learn concepts and procedures with hands-on learning activities. The construction of the learning activities required significant time and effort. On several occasions students did not complete scheduled learning activities during the classroom period and had to complete the activities outside of class. Use of hands-on learning activities through which students learn concepts and procedures is justifiable, even in view of the extra time and effort required, since the activities help students become self-directed learners. One of the most important functions of formal education must be to help students learn how to learn and thus become lifelong self-directed learners.

There are several advantages to the use of graphics calculators in the teaching and learning of college algebra. College algebra students are able to investigate complex mathematical models constructed from real-world data. Students are able to investigate certain topics traditionally relegated to the study of calculus. A discussion of rational functions in the treatment class taught by the researcher led to a discussion of limits. Using TI-81 Graphics Calculators students were able to

graph rational functions and determine the domains of the functions and the vertical and horizontal asymptotes. By storing values to the calculator memory and then evaluating the rational functions defined in the Y= list, students were able to investigate limits of the rational functions as the independent variables approached specific values and limits at infinity. Without graphics calculators this investigation of limits would have required tedious and time-consuming evaluations of the rational functions for several values of the independent variables.

Integration of graphics-calculator technology into the teaching and learning of mathematics could initiate reform in instructional strategies. Less emphasis could be placed on memorization of facts and procedures and paper-and-pencil skills, and more emphasis could be placed on conceptual understanding, multiple representations and connections, mathematical modeling, and mathematical problem formulation and solving.

Students need to learn to interpret information from graphs. Since graphs can be produced almost instantaneously by graphics-calculator technology, the emphasis can shift from producing graphs to interpretation of graphical representations. It is important that students learn to solve equations and inequalities and systems of equations and inequalities by standard paper-and-pencil algorithms and by graphical means. Some equations and inequalities are very difficult to solve by traditional paper-and-pencil algorithms, but solutions can be approximated to a high degree of precision with graphics-calculator technology.

Graphics calculators provide opportunities for students to connect graphical images, symbolic expressions, and sets of related numerical values. Students need to learn to represent functions numerically as tables of input-output pairs, symbolically as algebraic representations, and graphically as plots of input-output

points. Students need to translate across these representations and connect these representations to physical contexts.

Students also need to find connections between areas of mathematics. Algebra and geometry have been taught as isolated subjects, but there are numerous connections between the two areas. Students need to recognize, for example, that a real zero of a function corresponds to the x-intercept of a graph and that a real solution of a system of equations corresponds to a point of intersection of the graphs of the equations.

It is important that students learn to observe a phenomena involving a functional relationship between two variables, gather and plot observational data, fit a graph to the plotted points, use the graph to formulate a relationship between the two variables, and predict outcomes for unobserved values of one of the variables. Students can enter data into a graphics calculator, determine from a plotted scattergram the type of function the points seem to fit, and use the graphics calculator to create an equation for the curve of best fit. Students can then predict values of one of the variables when given values of the other variable.

This study produced mixed support for the use of the graphics calculator, but the graphics calculator is justifiable and a promising tool for the teaching and learning of mathematics. The graphics calculator holds promise as a key for genuine reform in mathematics education. Graphics-calculator technology requires mathematics teachers to rethink how and what they teach. The teacher's role must shift from dispensing information to facilitating learning. Less emphasis must be placed on procedures and manipulative skills and more emphasis on concepts and higher order thinking skills. Technology education must be provided for teachers as a component of preservice training and through inservice workshops. The results of this study compel additional studies for determining the

most effective methods for integrating graphics-calculator technology into mathematics education.

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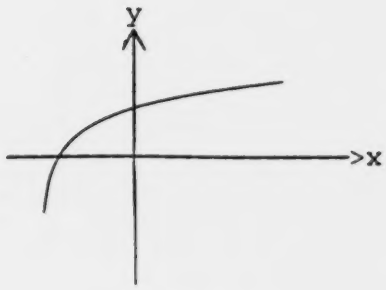
X- AND Y-INTERCEPTS

Consider the graph at the right. The point where the graph crosses the x-axis is called the x-intercept. What is the y-coordinate of this point?



The point where the graph crosses the y-axis is called the y-intercept. What is the x-coordinate of this point?

Definition of Intercepts

The point (a,0) is called an x-intercept of the graph of an equation if it is a solution point of the equation. To find an x-intercept, let y be zero and solve the equation for x.

The point (0,b) is called a y-intercept of the graph of an equation if it is a solution point of the equation. To find a y-intercept, let x be zero and solve the equation for y.

Find the x- and y-intercept(s) of the graph of $y = x^2 - 2x - 3$.

x-intercept(s) = _____

y-intercept(s) = _____

Press [RANGE] and use the cursor keys to set the range on your graphing calculator as follows:

Xmin = -4.7
Xmax = 4.8
Xscl = 1
Ymin = -6.2
Ymax = 6.4
Yscl = 1
Xres = 1

Graph the equation $y = x^2 - 2x - 3$ by pressing these keys in sequence:

[Y=] [X|T] [X^2] [-] [2] [X|T] [-] [3] [GRAPH]

When the graph has been plotted on the calculator screen, locate the intercepts by using [TRACE] and the cursor keys. When you press [TRACE], a cursor appears on the screen. By pressing [▶] several times in succession, you can move the cursor to the right along the curve. The x- and y-coordinates appear at the bottom of the screen.

What happens when you press [◄] ? Move the cursor along the curve to the location where the graph appears to cross the x-axis. The y-coordinate should be zero. The x-intercept is (,). Use the cursor keys to determine the coordinates of the other x-intercept. The other x-intercept is (,). The y-intercept is (,).

Find the x- and y-intercept(s) of the graph of $y = x^3 - 9x$.

x-intercept(s) = _____

y-intercept(s) = _____

Graph the equation $y = x^3 - 9x$. Press [Y=] and [CLEAR] to clear the previous equation. Then key in the current equation by pressing these keys in sequence.

[Y=] [X|T] [^] [3] [-] [9] [X|T]

Press [RANGE] and change the range on your graphing calculator so that Ymin = -12.4 and Ymax = 12.8. Press [GRAPH] to graph the equation. Use [TRACE] and the cursor keys to find the x- and y-intercept(s).

The coordinates of the intercepts of the graph of an equation may not be integers. Press [Y=] and clear the previous equation. Key in the equation $y = x^2 - 5$.

Press [ZOOM] and then [6]. This key sequence sets up the standard screen where both X and Y range from -10 to 10. Use [TRACE] and the cursor keys to move the cursor along the curve to a location where the graph appears to cross the x-axis. Observe the coordinates of this point.

Press [ZOOM] and then select [BOX] . A screen cursor appears. This cursor represents a corner of a rectangular box. We want to draw a box around the x-intercept. Use the cursor keys to move the cursor so that it is to the left and above the x-intercept. Press [ENTER] to set a corner of the box. Stretch the box horizontally and vertically by using the cursor keys until the x-intercept is inside the box. Press [ENTER] . The calculator graphs the portion of the graph inside the box.

Use [TRACE] and the cursor keys to move the cursor along the curve to the location where the graph appears to cross the x-axis. Note the coordinates of this point. Use [ZOOM] and [BOX] again to zoom in closer to the x-intercept. Repeat the procedure until you approximate the coordinates of the x-intecept to the desired degree of accuracy. To locate other intercepts, press [ZOOM] and [6] and repeat the above procedure.

Use your graphing calculator to approximate the x- and y-intercept(s) of the graphs of the following equations. You may need to adjust the range to get a complete graph of some of the equations.

1. $y = -3x + 6$

2. $y = -x^2 + 5x$

3. $y = x^3 + 3x^2 - x - 3$

4. $y = x^4 - 10x^2 + 16$

5. $y = (x + 2)(x - 3)(x + 6)$

6. $y = \sqrt{x} + 3$

7. $y = 2/(1 - x)$

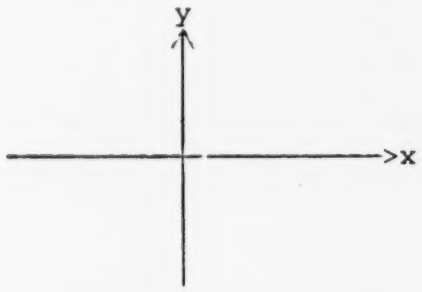
SYMMETRY

Set RANGE on your graphing calculator as follows:

Xmin = -9.4
Xmax = 9.6
Xscl = 1
Ymin = -12.4
Ymax = 12.8
Yscl = 2
Xres = 1

Graph the equation $y = x^2$.

Sketch the graph below, and state a geometric relationship between the y-axis and the graph.



Using TRACE , locate the point with x-coordinate 2. Complete the ordered pair (2,). Locate the point with x-coordinate -2. Complete the ordered pair (-2,).

Using TRACE , locate the appropriate points and complete the following ordered pairs.

(1,), (-1,)

(3,), (-3,)

(1.2,), (-1.2,)

(2.6,), (-2.6,)

(0.8,), (-0.8,)

Make a conjecture about the points on the graph of $y = x^2$.

The graph of $y = x^2$ is symmetric with respect to the y-axis.

A graph is symmetric with respect to the y-axis if the portion of the graph to the left of the y-axis is a mirror image of the portion to the right of the y-axis.

Complete the following statement: A graph is symmetric with respect to the y-axis if whenever (x,y) is on the graph, (,) is on the graph.

To test the graph of an equation for symmetry with respect to the y-axis, substitute -x for x in the equation. If this results in an equivalent equation, the graph is symmetric with respect to the y-axis.

Test the graph of $y = -2x^2$ for symmetry with respect to the y-axis. Verify your conclusion by graphing the equation.

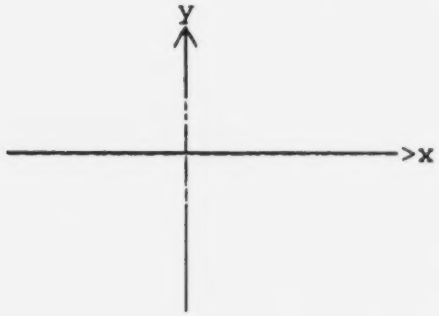
Set [RANGE] on your graphing calculator as follows:

Xmin = -9.4
Xmax = 9.6
Xscl = 1
Ymin = -3.1
Ymax = 3.2
Yscl = 1
Xres = 1

Graph the equation $x = y^2$.
First solve the equation for y: $y = \pm \sqrt{x}$.
Press the following keys in sequence:

[Y=] [$\sqrt{}$] [X|T] [▼] [-] [$\sqrt{}$] [X|T] [GRAPH]

Sketch the graph below and state a geometric relationship between the x-axis and the graph.



Using [TRACE] , locate the point with x-coordinate 4. Complete the ordered pair (4,). Press [▼] . Complete the ordered pair (4,).

Using [TRACE] , complete the following ordered pairs:

(,1) (,-1)

(,3) (,-3)

(1.2,) (1.2,)

(2.4,) (2.4,)

Make a conjecture about the points on the graph of $x = y^2$.

The graph of $x = y^2$ is symmetric with respect to the x-axis. A graph is symmetric with respect to the x-axis if the portion of the graph above the x-axis is a mirror image of the portion below the x-axis.

Complete the following statement: A graph is symmetric with respect to the x-axis if whenever (x,y) is on the graph, (,) is on the graph.

To test the graph of an equation for symmetry with respect to the x-axis, substitute -y for y in the equation. If this results in an equivalent equation, the graph is symmetric with respect to the x-axis.

Test the graph of $x^2 + y^2 = 25$ for symmetry with respect to the x-axis.

Verify your conclusion by graphing the equation. First solve the equation for y.

Set [RANGE] on your graphing calculator as follows:

Xmin = -4.7
Xmax = 4.8
Xscl = 1
Ymin = -12.4
Ymax = 12.8
Yscl = 1
Xres = 1

Graph the equation $y = x^3$.

Sketch the graph below, and state a geometric relationship between the origin and the graph.

Use TRACE to locate the point with x-coordinate 2. Complete the ordered pair (2,). Complete the ordered pair (-2,).

Use TRACE to locate the appropriate points and complete the following ordered pairs:

(1,) (-1,)

(0.5,) (-0.5,)

(1.2,) (-1.2,)

(0.03,) (-0.03,)

Make a conjecture about the points on the graph of $y = x^3$.

The graph of $y = x^3$ is symmetric with respect to the origin.

A graph is symmetric with respect to the origin if each point (x,y) on the graph has an image point (-x,-y) directly across the origin in the opposite quadrant.

To test the graph of an equation for symmetry with respect to the origin, substitute -x for x and -y for y in the equation. If this results in an equivalent equation, the graph is symmetric with respect to the origin.

Test the graph of $y = x^3 - x$ for symmetry with respect to the origin. Verify your conclusion by graphing the equation.

Test the following equations for symmetry. Verify your conclusions by graphing the equations.

1. $y = x^2 - x$

2. $y^2 + x - 2 = 0$

3. $y = x^4 - 4x^2$

4. $xy = 16$

5. $x^2 - 2y^2 = 12$

6. $y = |x - 2|$

7. $y = |x| + 3$

STRETCHES, SHRINKS, AND TRANSLATIONS OF GRAPHS

Graph the equation $y = x^2$. This graph is called a parabola. Notice that the graph is symmetric with respect to the y-axis. The y-axis is called a line of symmetry. The point (0,0) is the lowest point of the graph and is called the vertex of the parabola.

Graph the following equations on the same screen.

$y = x^2$, $y = .5x^2$, $y = 2x^2$, $y = 4x^2$

Which graphs lie below the graph of $y = x^2$?

Which graphs lie above the graph of $y = x^2$?

Propose an equation whose graph is between the graphs of $y = 4x^2$ and $y = 2x^2$. Support your guess graphically.

Describe the graph of $y = ax^2$ for various values of a.

Make a conjecture as to how these graphs change if a is negative.

If $a \geq 1$, the graph of $y = ax^2$ can be obtained from the graph of $y = x^2$ with a vertical stretch of the graph of $y = x^2$ by a factor of a.

If $0 \leq a < 1$, the graph of $y = ax^2$ can be obtained from the graph of $y = x^2$ with a vertical shrink of the graph of $y = x^2$ by a factor of a.

The graph of $y = ax^2$ has vertex (0,0), and the y-axis is a line of symmetry.

Graph the following equations on the same screen.

$y = -x^2$, $y = -3x^2$, $y = -0.5x^2$

How do these graphs differ from the previous graphs?

Describe the graph of $y = ax^2$ for various negative values of a.

Transformations of Stretching/Shrinking and Reflection

Condition on coefficient a	To obtain $y = ax^2$ from $y = x^2$
$a > 1$	Stretch by a factor of a
$0 < a < 1$	Shrink by a factor of a
$-1 < a < 0$	Shrink by factor $\|a\|$ and reflect across the x-axis
$a < -1$	Stretch by factor $\|a\|$ and reflect across the x-axis

---

Graph the following equations on the same screen.

$y = x^2$, $y = x^2 + 4$, $y = x^2 - 3$

Describe the graph of $y = x^2 + k$ for various values of k. Where is the vertex located? What is the line of symmetry?

The graph of $y = x^2 + k$ can be obtained from the graph of $y = x^2$ by a vertical shift.

If $k > 0$, the shift is up k units.

If $k < 0$, the shift is down $|k|$ units.

For any k, the graph of $y = x^2 + k$ has the vertex $(0,k)$ and the y-axis is the line of symmetry.

Graph the following equations on the same screen.

$y = x^2$, $y = (x - 3)^2$, $y = (x + 4)^2$

Describe the graph of $y = (x - h)^2$ for various values of h. Where is the vertex located? What is the line of symmetry?

The graph of $y = (x - h)^2$ can be obtained from the graph of $y = x^2$ by a horizontal shift.

If $h > 0$, the shift is right h units.

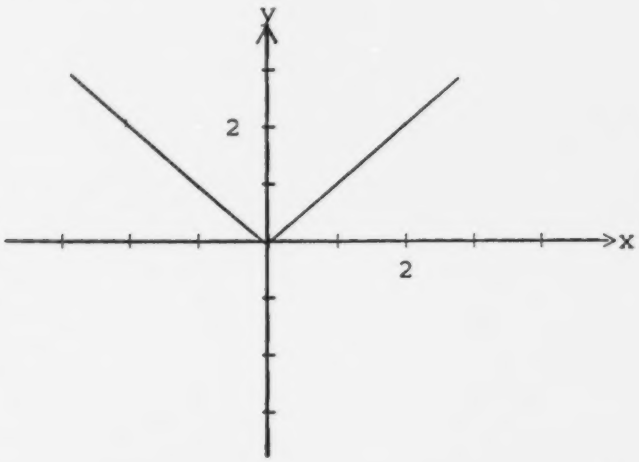
If $h < 0$, the shift is left $|h|$ units.

For any h, the graph of $y = (x - h)^2$ has the vertex (h,0) and the line of symmetry $x = h$.

Describe how the graph of $y = (x - 5)^2$ can be obtained from the graph of $y = x^2$. Test your conjecture graphically.

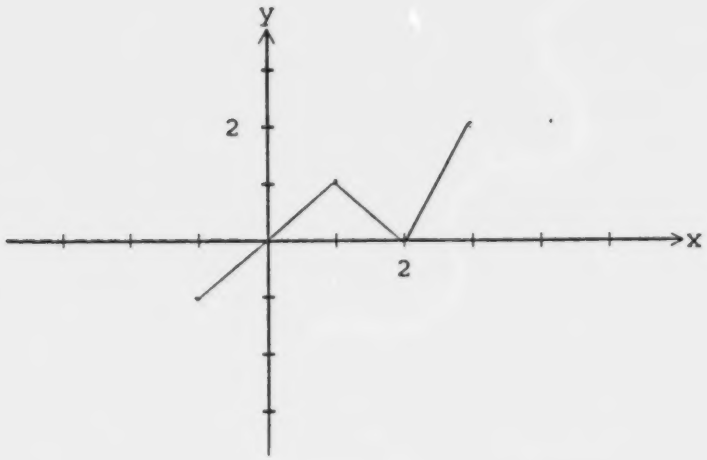
The graph below is the graph of $y = |x|$. Sketch and label on the same set of axes graphs of the following equations.

$y = |x| + 2$, $y = |x - 2|$, $y = |x + 1| - 1$



Given the graph of f(x) below, sketch and label graphs of the following:

$g(x) = f(x) + 1$, $h(x) = f(x - 1)$, $k(x) = f(x + 2) - 2$



GRAPHING FUNCTIONS AND FINDING DOMAINS AND RANGES

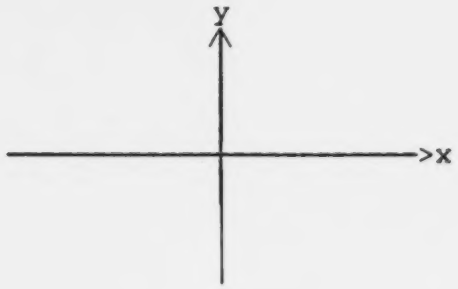
The TI-81 Graphics Calculator is a function grapher. We can determine the domain and the range of a function by using the graphing features of the TI-81. Consider the function $f(x) = x^2 - 6$. The domain is all real numbers. Since $x^2 \geq 0$ for all real values of x, $x^2 - 6 \geq -6$; the range of f is all real numbers y such that $y \geq -6$.

Set [RANGE] on your graphing calculator as follows:

Xmin = -4.7
Xmax = 4.8
Xscl = 1
Ymin = -9.3
Ymax = 9.6
Yscl = 1
Xres = 1

Graph the function $f(x) = x^2 - 6$. Use [TRACE] to locate the lowest point on the graph. Verify that the range of the function is $[-6,\infty)$.

Graph the function $f(x) = |x^2 - 6|$. Sketch the graph below.

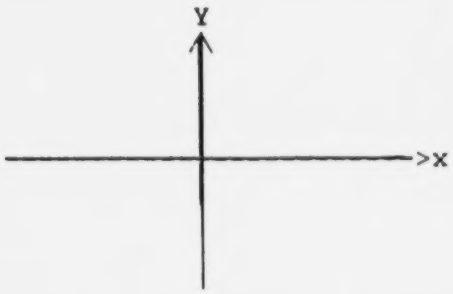


What is the domain?

What is the range?

Consider the function $f(x) = \sqrt{x - 4}$. Since $x - 4$ must be greater than or equal to zero, the domain of the function is all real numbers such that $x \geq 4$. Since $\sqrt{x - 4}$ is greater than or equal to zero for all real values of x greater than or equal to four, the range is all real numbers y such that $y \geq 0$. Graph the function, and verify the domain and range.

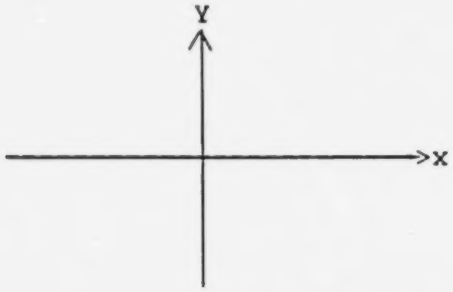
Consider the function $f(x) = \sqrt{x^2 - 4}$. Press [ZOOM] [6] to graph the function on the standard screen. Graph the function, and sketch the graph below.



What is the domain of the function?

What is the range of the function?

Graph the relation $y^2 = x$. First solve the equation for y so that $y = \pm\sqrt{x}$. Set Y1 = $\sqrt{x}$ and Y2 = $-\sqrt{x}$. Sketch the graph below.



What is the domain?

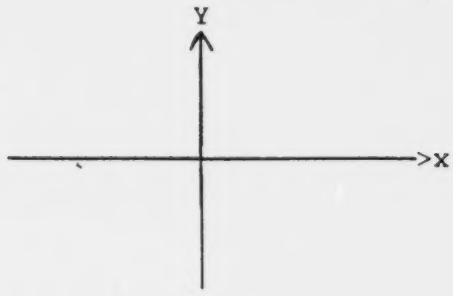
What is the range?

By the definition of a function, at most one y-value corresponds to a given x-value. It follows that a vertical line can intersect the graph of a function at most once. Does the graph above represent y as a function of x? Why or why not?

Set [RANGE] as follows:

Xmin = -10
Xmax = 10
Xscl = 1
Ymin = -2
Ymax = 2
Yscl = 1
Xres = 1

Graph the function $f(x) = \sqrt{x}/(2x - 4)$. Sketch the graph below.



What is the domain? (Consider any restrictions on x)

What is the range?

The greatest integer function is built into the TI-81. It is denoted by ⟦x⟧ or Int x. Int x is defined to be the greatest integer less than or equal to x. For example, ⟦1.2⟧ = 1, ⟦2⟧ = 2, and ⟦-3.6⟧ = -4.
Graph f(x) = ⟦x⟧ on the standard screen and find the domain and range.

Use the following key squence:

Press [MODE] and set your calculator in the dot mode.
Press [ENTER]. Press [Y=] and then [MATH]. Move the cursor to [NUM] and then press [4]. Press [X|T] and [ZOOM] [6].

What is the domain of the function?

Use [TRACE] to find the range of the function. What is the range of the function?

Set [RANGE] as follows:

Xmin = -4.7
Xmax = 4.8
Xscl = 1
Ymin = -3.1
Ymax = 3.2
Yscl = 1
Xres = 1

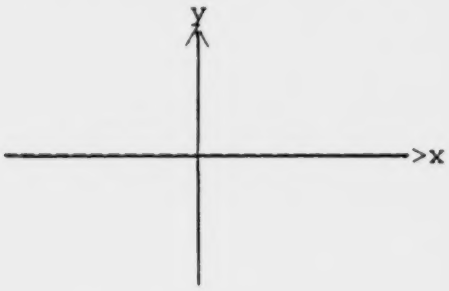
Graph the following piece-wise function, and sketch the graph below.

$$f(x) = \begin{cases} x^2 \text{ if } x \leq 0 \\ 2x \text{ if } x \geq 1 \end{cases}$$

Enter the function into your calculator as follows:

$Y1 = x^2(x \leq 0) + 2x(x \geq 1)$

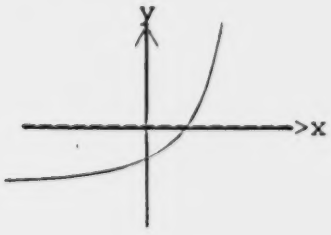
The symbols $\leq$ and $\geq$ may be found by pressing [2nd] [MATH] .



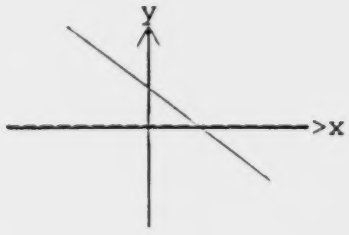
What is the domain?

What is the range?

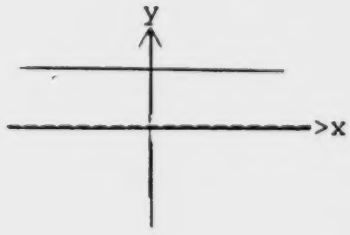
Consider the graphs below.



(a)



(b)



(c)

In figure (a), the function values increase (the graph rises) as the values of x increase (vary from left to right); that is, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

In figure (b), the function values decrease (the graph falls) as the values of x increase (vary from left to right); that is, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

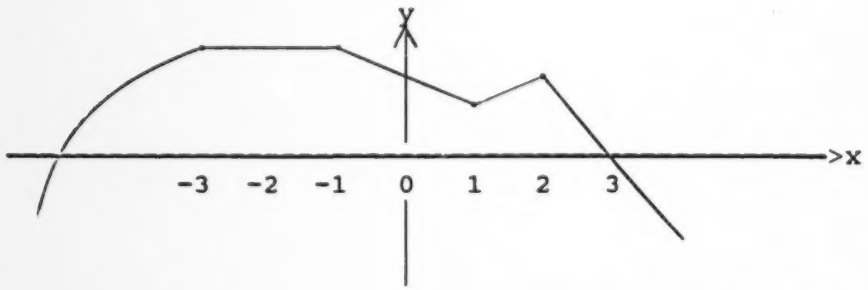
In figure (c), the function values remain constant (the graph neither rises nor falls) as the values of x increase (vary from left to right).

A function f is increasing on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies that $f(x_1) < f(x_2)$.

A function f is decreasing on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies that $f(x_1) > f(x_2)$.

A function f is constant on an interval if, for any x_1 and x_2 in the interval, $f(x_1) = f(x_2)$.

Given the graph below, determine the intervals on which the function is increasing, decreasing, or constant.



The function is increasing on the intervals $(-\infty, -3)$ and $(1,2)$.
The function is decreasing on the invervals $(-1,1)$ and $(2,\infty)$.
The function is constant on the interval $(-3,-1)$.

Graph the following functions. Using TRACE , estimate the intervals on which the functions are increasing, decreasing, or constant. Also, estimate the domain and range of each function.

1. $f(x) = x^2$
2. $f(x) = -2x^2 - 4x + 7$
3. $f(x) = |x + 6|$
4. $f(x) = x^3 - 25x$
5. $f(x) = x^3 + x$
6. $f(x) = |x + 2| + |x - 2|$
7. $f(x) = \sqrt{x^2 - 16}$
8. $f(x) = -2x + 7$

POLYNOMIAL FUNCTIONS

A polynomial function is one that can be written in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where n is a nonnegative integer and the coefficients $a_0, a_1, \ldots, a_n$ are real numbers. If $a_n \neq 0$, then n is the degree of the polynomial function.

Is the function $f(x) = x^3 + 2x^2 - 5x - 6$ a polynomial function?

What is its degree?

Graph $f(x) = x^3 + 2x^2 - 5x - 6$ on the standard viewing screen.

Notice that the graph of f(x) falls to the left and rises to the right.

What is the value of n? Is n even or odd?

What is the value of a_n? Is a_n positive or negative?

For each of the following functions,

(a) graph the function

(b) determine whether n is even or odd

(c) determine whether a_n is positive or negative

(d) discuss how the graph falls and rises

1. $g(x) = -x^3 - 2x^2 + 8x + 2.$

2. $h(x) = -2x^5 + 4x^2 - 5.$

3. $k(x) = 2x^5 - 4x^2 + 5.$

4. $f(x) = 2x^2 - x - 7.$

5. $g(x) = -2x^2 + 5x + 3.$

6. $h(x) = x^4 - 3x^2 + 2.$

7. $k(x) = -2x^4 + 8x^2 - 2.$

Write some conjectures about the relationship between n, a_n, and how the graph of a polynomial function rises and falls.

The graph of the polynomial function

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

falls and rises in the following manner:

If n is odd and $a_n > 0$, the graph falls to the left and rises to the right.

If n is odd and $a_n < 0$, the graph rises to the left and falls to the right.

If n is even and $a_n > 0$, the graph rises to the left and right.

If n is even and $a_n < 0$, the graph falls to the left and right.

Discuss the left and right behavior of the graphs of the following equations. Graph the functions to verify your conclusions.

1. $f(x) = -x^5 - x - 1$
2. $g(x) = .4x^4 - x^3 - 3$
3. $h(x) = x^3 - 2x^2 + x - 1$
4. $k(x) = -.2x^6 - 2x^2 - 2$

Graph $f(x) = x^3 + 2x^2 - 5x - 6$ on the standard viewing screen.

How many times does the graph cross the x-axis?

What is true about the y-coordinates of the points where the graph crosses the x-axis?

Using [TRACE] and [ZOOM], estimate the x-coordinates of the points where the graph crosses the x-axis.

Show that $(x - 2)(x + 1)(x + 3) = x^3 + 2x^2 - 5x - 6$.

What are the exact solutions to the equation $x^3 + 2x^2 - 5x - 6 = 0$?

How do the x-coordinates of the points where the graph crosses the x-axis compare with the exact solutions of the equation $x^3 + 2x^2 - 5x - 6 = 0$?

What are the zeros of $f(x) = x^3 + 2x^2 - 5x - 6$?

What are the x-intercepts of the graph of $f(x) = x^3 + 2x^2 - 5x - 6$?

Write some conjectures about the coordinates of the points where the graph crosses the x-axis.

If f is a polynomial function and a is a real number, the following statements are equivalent.

1. a is a zero of the function f.
2. (a,0) is an x-intercept of the graph of f.
3. $x = a$ is a solution of the polynomial equation $f(x) = 0$.
4. $(x - a)$ is a factor of the polynomial $f(x)$.

Given $f(x) = x^3 - 3x^2 - 4x$, find the factors of $f(x)$, the solutions of the equation $f(x) = 0$, the x-intercepts of the graph of f, and the zeros of f.

REAL ZEROS OF POLYNOMIAL FUNCTIONS

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of positive real zeros of f is either equal to the number of variations in sign of f(x) or is less than that number by an even integer.
2. The number of negative real zeros of f is either equal to the number of variations in sign of f(-x) or is less than that number by an even integer.

Use Descartes's Rule of Signs to determine the possible number of positive and negative real zeros of $f(x) = x^3 - x^2 - 9x + 9$.

The polynomial $f(x) = \overset{+ \text{ to } -}{x^3} - \overset{- \text{ to } +}{x^2} - 9x + 9$ has two variations in sign.

$f(-x) = -x^3 - \overset{- \text{ to } +}{x^2} + 9x + 9$ has one variation in sign.

The polynomial $f(x) = x^3 - x^2 - 9x + 9$ has either two or no positive real zeros and one negative real zero.

Verify this by graphing f.

For each of the following polynomial functions, determine the possible number of positive and negative real zeros of the function. Verify your findings by graphing each of the functions.

1. $f(x) = x^4 - x^3 + x^2 - 6$
2. $g(x) = x^5 - x^4 - 2x^3 + x^2 - 5x + 2$
3. $h(x) = 2x^3 - 8x^2 - x + 5$
4. $k(x) = x^3 - x + 5$

The Rational Zero Test

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients, then every rational zero of f has the form

Rational zero = p/q

where p and q have no common factors other than 1 and

p = a factor of the constant term a_0

q = a factor of the leading coefficient a_n.

Find the rational zeros of $f(x) = 4x^3 - 3x^2 - 4x + 3$.

Factors of p are ± 1, ± 3

Factors of q are ± 1, ± 2, ± 4

Possible rational zeros are ± 1, ± 3, $\pm 1/2$, $\pm 1/4$, $\pm 3/2$, $\pm 3/4$

By synthetic division we determine that 3/4 is a zero.

3/4	4	-3	-4	3
		3	0	-3
	4	0	-4	0

Therefore, $f(x) = (x - 3/4)(4x^2 - 4) = (x - 3/4)(4)(x - 1)(x + 1)$, and the zeros of f are $x = 3/4$, $x = 1$, and $x = -1$.

Find all the real zeros of $f(x) = 2x^3 - 5x^2 + x + 3$.

Factors of p are ± 1, ± 3

Factors of q are ± 1, ± 2

Possible rational zeros are ± 1, ± 3, $\pm 1/2$, $\pm 3/2$

By synthetic division we determine that 3/2 is a zero.

3/2	2	-5	1	3
		3	-3	-3
	2	-2	-2	0

$f(x) = (x - 3/2)(2x^2 - 2x - 2) = (x - 3/2)(2)(x^2 - x - 1)$

Using the quadratic formula, we find the two additional zeros.

$x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$x = \frac{1 + \sqrt{5}}{2} = 1.618 \quad \text{or} \quad x = \frac{1 - \sqrt{5}}{2} = -0.618$$

The real zeros of f are $x = 3/2$, $x = 1.618$, and $x = -0.618$.

Find all real zeros of $f(x) = x^4 + x^3 - 8x^2 + 8$.

Factors of p are ± 1, ± 2, ± 4, ± 8

Factors of q are ± 1

Possible rational zeros are ± 1, ± 2, ± 4, ± 8

Graph f(x). Can you use the graph of f to eliminate some of the possible rational zeros?

If so, which of the possible rational zeros look like reasonable choices for zeros of f?

By synthetic division, we determine that -1 and 2 are zeros.

-1	1	1	-8	0	8
		-1	0	8	-8
	1	0	-8	8	0

2	1	0	-8	8
		2	4	-8
	1	2	-4	0

The factors of f(x) are $f(x) = (x + 1)(x - 2)(x^2 + 2x - 4)$.

Using the quadratic formula, we find the additional zeros.

$x^2 + 2x - 4 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-4)}}{2(1)} = \frac{-2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$$

$x = 1 + \sqrt{5} = 2.236$ or $x = 1 - \sqrt{5} = -1.236$

The real zeros of f are $x = -1$, $x = 2$, $x = 2.236$, and $x = -1.236$

Verify from the graph of f that the graph crosses the x-axis at the four zeros.

Find all real zeros of $f(x) = x^5 - 2x^4 - 4x^3 + 7x^2 + 4x - 4$.

$f(-x) = -x^5 - 2x^4 + 4x^3 + 7x^2 - 4x - 4$

Descartes's Rule of Signs indicates three or one positive real zeros and two or no negative real zeros.

Factors of p are ± 1, ± 2, ± 4

Factors of q are ± 1

Possible rational zeros are ± 1, ± 2, ± 4

Synthetic division produces the following:

2	1	-2	-4	7	4	-4
		2	0	-8	-2	4
	1	0	-4	-1	2	0

2	1	0	-4	-1	2
		2	4	0	-2
	1	2	0	-1	0

-1	1	2	0	-1
		-2	-1	1
	1	1	-1	0

Therefore $f(x) = x^5 - 2x^4 - 4x^3 + 7x^2 + 4x - 4$

$= (x - 2)(x - 2)(x + 1)(x^2 + x - 1)$

which gives us the following five real zeros: 2, 2, -1, 0.618, -1.618.

Does this agree with the findings from Descartes's Rule of Signs?

Graph $f(x) = x^5 - 2x^4 - 4x^3 + 7x^2 + 4x - 4$.

Note from the graph that the real zeros appear as x-intercepts. The graph crosses the x-axis at three zeros and touches the x-axis at $x = 2$, the root of multiplicity two.

Find all real zeros for each of the following polynomial functions. Use Descartes's Rule of Signs to determine the number of possible positive and negative real roots and the rational zero test to find the rational zeros. Graph the functions to verify your findings.

1. $f(x) = x^3 - 3x - 2$

2. $f(x) = 3x^3 - 8x^2 + x + 2$

3. $f(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$

RATIONAL FUNCTIONS

A rational function is one that can be written in the form $f(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomial functions, $Q(x) \neq 0$.

The domain of a rational function is the set of all real numbers except those for which the denominator is 0.

The domain of $f(x) = 1/x$ is all real numbers except 0.

The domain of $g(x) = \dfrac{3x^2 - x + 4}{(x - 2)(x + 3)}$ is all real numbers except 2 and -3.

The domain of $h(x) = \dfrac{x + 2}{x^2 - 4}$ is all real numbers except 2 and -2.

Set [RANGE] as follows:

Xmin = -10
Xmax = 10
Xscl = 1
Ymin = -2
Ymax = 2
Yscl = 1
Xres = 1

Graph $f(x) = 1/x$. Observe the values of the x- and y-coordinates as you use the trace key to move the cursor along the curve. As the cursor moves to the left (as x gets smaller), does the y-coordinate seem to approach a certain value? If so, what is this value?

As the cursor moves to the right (as x increases), does the y-coordinate seem to approach a certain value? If so, what is this value?

Set [RANGE] as follows:

Xmin = -100
Xmax = 100
Xscl = 1
Ymin = -.1
Ymax = .1
Yscl = 1
Xres = 1

Graph $f(x) = 1/x$. Use the trace key to move the cursor along the curve, and repeat the above observations.

What conclusions do you make about the x- and y-coordinates as the cursor moves along the curve?

Set [RANGE] as follows:

Xmin = -1000
Xmax = 1000
Xscl = 1
Ymin = -.01
Ymax = .01
Yscl = 1
Xres = 1

Graph f(x) = 1/x, and repeat the above observations.

Write your conclusions.

For the function f(x) = 1/x, we say that as x approaches positive infinity or negative infinity, f(x) approaches 0. We write

as $x \longrightarrow +\infty$ or as $x \longrightarrow -\infty$, $f(x) \longrightarrow 0$.

The horizontal line y = 0 is called a horizontal asymptote of f(x) = 1/x.

Definition: The line y = b is called a horizontal asymptote of the graph of f if

$f(x) \longrightarrow b$ as $x \longrightarrow +\infty$ or $x \longrightarrow -\infty$.

Graph $g(x) = \dfrac{2x + 1}{x + 1}$ and determine the horizontal asymptote.

Graph $h(x) = \dfrac{x^2 - 5}{x^2 + 1}$ and determine the horizontal asymptote.

Let f be the rational function given by

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}.$$

If $n < m$, then the graph of f has the x-axis as a horizontal asymptote.

If $n = m$, then the graph of f has the line $y = a_n/b_m$ as a horizontal asymptote.

If $n > m$, then the graph of f has no horizontal asymptote.

Find the horizontal asymptotes of the following rational functions, and verify your findings by graphing the functions.

1. $f(x) = \dfrac{3x^2 - 2x + 7}{x^2 + 4x + 1}$

2. $g(x) = \dfrac{5x + 4}{x^3 - 3}$

3. $h(x) = \dfrac{-2x - 3}{x + 2}$

Set [RANGE] as follows:

Xmin = -1.5
Xmax = 6
Xscl = 1
Ymin = -10
Ymax = 10
Yscl = 1
Xres = 1

Set Y1 = f(x) = 2/(x - 3) and graph the function. Use the trace key to investigate the values of the y-coordinates as x approaches 3 from the left and from the right.

Complete the table below by storing each given value in X and then evaluating Y1 at this value.

x	f(x)	x	f(x)
2.0		4.0	
2.9		3.1	
2.99		3.01	
2.999		3.001	
2.9999		3.0001	
2.99999		3.00001	

It appears from the graph and from the table that as x approaches 3 from the left ($x \longrightarrow 3^-$), f(x) approaches negative infinity, and as x approaches 3 from the right ($x \longrightarrow 3^+$), f(x) approaches positive infinity. We write:

As $x \longrightarrow 3^-$, $f(x) \longrightarrow -\infty$, and as $x \longrightarrow 3^+$, $f(x) \longrightarrow +\infty$.

The line x = 3 is called a vertical asymptote of f(x) = 2/(x - 3).

Definition. The line x = a is called a vertical asymptote of f if

$f(x) \longrightarrow +\infty$ or $f(x) \longrightarrow -\infty$

as $x \longrightarrow a$ from the right or from the left.

If the numerator and denominator of a rational function have no common factors, a vertical asymptote will occur when the denominator is zero.

Consider f(x) = 5/(x - 4). The denominator is zero if x - 4 = 0 or x = 4. The vertical line x = 4 is a vertical asymptote for f. Graph f and verify that x = 4 is a vertical asymptote.

Consider $g(x) = \dfrac{x + 2}{x^2 - 16}$. The denominator is zero if $x^2 - 16 = 0$ or $(x - 4)(x + 4) = 0$ or x = 4 or x = -4. The vertical lines x = 4 and x = -4 are vertical asymptotes for g.

Graph g and verify that x = 4 and x = -4 are vertical asymptotes.

Find the vertical asymptotes of each of the following rational functions, and verify your findings by graphing the functions.

1. $f(x) = 1/(x + 2)$

2. $g(x) = \dfrac{x - 1}{x^2 - 9}$

3. $h(x) = \dfrac{12}{x^2 - 3x - 4}$

EXPONENTIAL FUNCTIONS

Definition: The exponential function f with base a is denoted by $f(x) = a^x$ where $a > 0$, $a \neq 1$, and x is any real number.

Set [RANGE] as follows:

Xmin = -5
Xmax = 2
Xscl = 1
Ymin = 0
Ymax = 2
Yscl = 1
Xres = 1

Graph the following functions on the same set of axes:

$f(x) = 2^x$, $g(x) = 3^x$, $h(x) = 5^x$

For what values of x is $2^x > 3^x > 5^x$?

For what values of x is $2^x < 3^x < 5^x$?

For what value of x is $2^x = 3^x = 5^x$?

Set [RANGE] as follows:

Xmin = -2
Xmax = 5
Xscl = 1
Ymin = 0
Ymax = 2
Yscl = 1
Xres = 1

Graph $f(x) = (1/2)^x$, $g(x) = (1/3)^x$, and $h(x) = (1/5)^x$ on the same set of axes:

For what values of x is $(1/2)^x > (1/3)^x > (1/5)^x$?

For what values of x is $(1/2)^x < (1/3)^x < (1/5)^x$?

For what value of x is $(1/2)^x = (1/3)^x = (1/5)^x$?

For $f(x) = a^x$, $a > 1$,

What is the domain?

What is the range?

What is the intercept?

Is the function an increasing or decreasing function?

What is the horizontal asymptote?

For $f(x) = a^x$, $0 < a < 1$,

What is the domain?

What is the range?

What is the intercept?

Is the function increasing or decreasing?

What is the horizontal asymptote?

Consider the function $f(x) = (1 + 1/x)^x$.

Since the base of an exponential function must be positive, $1 + 1/x > 0$.

$1 + 1/x > 0$ if and only if $x < -1$ or $x > 0$.

Set $Y1 = (1 + 1/x)^x$ and press [ZOOM] [6].

Does it appear that the graph of the function has a horizontal asymptote?

Recall that the line $y = b$ is called a horizontal asymptote of the graph of f if $f(x) \longrightarrow b$ as $x \longrightarrow +\infty$ or $x \longrightarrow -\infty$.

Complete the table below by storing each given value in X and then evaluating Y1 at the given value.

x	f(x)
1	
10	
100	
1,000	
10,000	
100,000	
1,000,000	
10,000,000	
100,000,000	
1,000,000,000	

x	f(x)
-1	
-10	
-100	
-1,000	
-10,000	
-100,000	
-1,000,000	
-10,000,000	
-100,000,000	
-1,000,000,000	

The horizontal asymptote of $f(x) = (1 + 1/x)^x$ is the line $y = k$ where k is an irrational number that can be approximated as $k = 2.718281828$. This irrational number is denoted by e and is called the natural base. The function $f(x) = e^x$ is called the natural exponential function.

The natural exponential function can be evaluated and graphed with the TI-81 Graphics Calculator.

Evaluate e^2. Press [e^x] [2] [ENTER].

Graph $f(x) = e^x$. Press [Y=] [e^x] [X|T] [ZOOM] [6].

Suppose the population P of a certain country is increasing at a constant rate r each year where r is in decimal form.

$P + Pr = P(1 + r)$ = population one year later

$P(1 + r) + P(1 + r)r = P(1 + r)(1 + r) = P(1 + r)^2$ = population two years later

$P(1 + r)^3$ = population three years later

.
.
.

$P(1 + r)^t$ = population t years later

This is an example of an exponential growth function.

A town has a population of 25,000 and the population is increasing at the rate of 3.4% per year.

The function $f(t) = 25{,}000(1 + .034)^t$ gives the population t years later.

Set [RANGE] as follows:

Xmin = 0
Xmax = 25
Xscl = 1
Ymin = 0
Ymax = 60,000
Yscl = 1
Xres = 1

Set Y1 = $25{,}000(1.034)^{X}$, set Y2 = 50,000, and press [GRAPH].

Use [TRACE] to estimate where the graphs of Y1 and Y2 intersect. Estimate to the nearest year how long it will take the population to reach 50,000.

Graph the following functions on the same set of axes:

$f(x) = 2^{x}$, $g(x) = 2^{2x}$, $h(x) = 2^{3x}$

If $f(x) = a^{bx}$, what is the effect of b on the graph of the function?

Graph the following functions on the same set of axes:

$f(x) = 2^{x}$, $g(x) = 2^{x + 3}$, $h(x) = 2^{x - 4}$

If $f(x) = a^{bx + c}$, what is the effect of c on the graph of the function?

LOGARITHMIC FUNCTIONS

Every function of the form $f(x) = a^x$ passes the horizontal line test and therefore has an inverse. The inverse of the exponential function with base a is called the logarithmic function with base a.

Definition: For $x > 0$ and $0 < a \neq 1$, $y = \log_a x$ if and only if $x = a^y$. The function given by $f(x) = \log_a x$ is called the logarithmic function with base a.

$\log_3 9 = 2$ since $3^2 = 9$ $\qquad$ $\log_4 2 = 1/2$ since $4^{1/2} = 2$

$\log_{10} 100 = 2$ since $10^2 = 100$ $\qquad$ $\log_{10}(.01) = -2$ since $10^{-2} = .01$

$\log_5 1 = 0$ since $5^0 = 1$ $\qquad$ $\log_2 2 = 1$ since $2^1 = 2$

Properties of logarithms.

$\log_a a = 1$ since $a^1 = a$

$\log_a(a^x) = x$ since $a^x = a^x$

If $\log_a x = \log_a y$, then $x = y$

Graph $y = 2^x$. Sketch the graph below. Sketch the line $y = x$. Since $y = \log_2 x$ is the inverse of $y = 2^x$, the graph of $y = \log_2 x$ is the reflection of the graph of $y = 2^x$ in the line $y = x$. Use this fact to sketch the graph of $y = \log_2 x$.

The logarithmic function with base 10 is called the common logarithmic function and is denoted on the TI-81 Graphics Calculator by log .

To evaluate $\log_{10} 100$ press [log] [100] [ENTER] . $\log_{10} 100 = 2$

To evaluate $\log_{10} 20$ press [log] [20] [ENTER] . $\log_{10} 20 = 1.301029996$

Set [RANGE] as follows:

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -5
Ymax = 5
Yscl = 1
Xres = 1

To graph $f(x) = \log_{10} X$ press [Y1] [log] [X|T] [GRAPH].

Set $Y2 = 10^X$ and $Y3 = x$.

Graph Y1, Y2, and Y3 on the same screen. Do the graphs of Y1 and Y2 appear to be reflections in the line $y = x$?

Remember that $y = \log_{10} x$ is the inverse of $y = 10^x$.

The most widely used base for logarithmic functions is e. The logarithmic function with base e is called the natural logarithmic function and is denoted by the symbol ln x.

The function defined by $f(x) = \log_e x = \ln x$, $x > 0$, is called the natural logarithmic function.

Properties of natural logarithms.

$\ln 1 = 0$ since $e^0 = 1$

$\ln e = 1$ since $e^1 = e$

$\ln e^x = x$ since $e^x = e^x$

If $\ln x = \ln y$, then $x = y$

To evaluate ln 2, press [ln] [2] [ENTER].

Set $Y1 = \ln x$, $Y2 = e^x$, and $y3 = x$. Graph Y1, Y2, and Y3 on the same screen. Do the graphs of Y1 and Y2 appear to be reflections in the line $y = x$?

Consider the graphs of $y = \log_2 x$, $y = \log x$, and $y = \ln x$. All of these are of the form $y = \log_a x$, $a > 1$.

For the graph of $y = \log_a x$, $a > 1$

What is the domain?

What is the range?

What is the intercept?

Is the function increasing or dereasing?

What is a vertical asymptote?

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$.

Then $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Since the TI-81 Graphics Calculator has a key for common logarithms [log] and a key for natural logarithms [ln] , we use these to evaluate logarithms to other bases.

$$\log_5 15 = \frac{\log 15}{\log 5} = \frac{\ln 15}{\ln 5} = 1.682606194$$

$$\log_2 12 = \frac{\log 12}{\log 2} = \frac{\ln 12}{\ln 2} = 3.584962501$$

To graph $f(x) = \log_4 x$, set Y1 = (log x/log 4) or Y1 = (ln x/ln 4).

Set $Y1 = 2^x$, $Y2 = \log_2 x$ = (log x/log 2), and Y3 = x. Graph all three functions on the same screen. Do the graphs of Y1 and Y2 appear to be reflections in the line y = x?

Set [RANGE] as follows:

Xmin = 0
Xmax = 100
X scl = 1
Ymin = -1
Ymax = 5
Yscl = 1
Xres = 1

Set Y1 = log x, Y2 = log 10x, and Y3 = log 100x. Graph all three functions on the same screen. Use [TRACE] and the cursor keys to move among the curves and observe the change in the y-coordinates as you move from one curve to another. What do you observe?

Set [RANGE] as follows:

Xmin = 0
Xmax = 10
Xscl = 1
Ymin = -1
Ymax = 2
Yscl = 1
Xres = 1

Set Y1 = log x, Y2 = log 5x, and Y3 = Y2 - Y1. Graph all three functions on the same screen. Describe the graph of Y3.

Set [RANGE] as follows:

Xmin = 0
Xmax = 10
Xscl = 1
Ymin = -1
Ymax = 2
Yscl = 1
Xres = 1

Set Y1 = log x + log 5 and Y2 = log 5x. Graph these two functions on the same screen. What do you observe?

Set Y3 = Y2/Y1. Graph Y1, Y2, and Y3 on the same screen. Use TRACE and the cursor keys to trace along the graph of Y3. What do all the points of Y3 have in common?

Write a conjecture about the relationship between the graphs of $y = \log x + \log A$ and $y = \log Ax$ where A is any positive number.

Set Y1 = 2 log x, Y2 = $\log x^2$, and Y3 = Y2/Y1. Graph the three functions on the same screen. Use TRACE and the cursor keys to trace along the graph of Y3. What do all the points of the graph of Y3 have in common?

Set Y1 = 3 log x, Y2 = $\log x^3$, and Y3 = Y2/Y1. Graph the three functions on the same screen. Use TRACE and the cursor keys to trace along the graph of Y3. What do all the points of the graph of Y3 have in common?

Write a conjecture about the relationship between the graphs of $y = n \log x$ and $y = \log x^n$, where n is a positive integer. Test your conjecture for several values of n.